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Asymptotics of High Order Noise Corrections

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We consider an evolution operator for a discrete Langevin equation with a strongly hyperbolic classical dynamics and noise with finite moments. Using a perturbative expansion of the evolution operator we calculate high order corrections to its trace in the case of a quartic map and Gaussian noise. The asymptotic behaviour is investigated and is found to be independent up to a multiplicative constant of the distribution of noise.

KEY WORDS: Discrete Langevin equation; weak noise; high order asymptotics.

1. INTRODUCTION

In natural phenomena stochastic processes of various strength always have an influence. In the three preceding papers[(1–3)] the effects of noise on measurable properties such as dynamical averages[(4)] in classical chaotic dynamical systems were systematically accounted. In order to construct a statistical theory of dynamical systems one introduces densities of particles governed by a corresponding evolution operator. For a repeller the leading eigenvalue of this operator $\mathscr{L}$ yields a physically measurable property of the dynamical system, the escape rate from the repeller. In the case of deterministic flows, the periodic orbit theory yields explicit and numerically efficient formulas for the spectrum of $\mathscr{L}$ as zeros of its spectral determinant.[(6)]

The theory developed is closely related to the semiclassical expansions based on Gutzwiller's formula for the trace in terms of classical periodic orbits[(5)] in that both are perturbative theories (in the noise strength or $\hbar$)

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derived from saddle point expansions of a path integral containing a dense set of unstable stationary points (typically periodic orbits). The analogy with quantum mechanics and field theory is made explicit in ref. 1 where Feynman diagrams are used to find the lowest nontrivial noise corrections to the escape rate.

An elegant method, inspired by the classical perturbation theory of celestial mechanics, is that of smooth conjugations.[2] In this approach the neighbourhood of each saddle point is flattened by an appropriate coordinate transformation, so the focus shifts from the original dynamics to the properties of the transformations involved. The expressions obtained for perturbative corrections in this approach are much simpler than those found from the equivalent Feynman diagrams. Using these techniques, we were able to extend the stochastic perturbation theory to the fourth order in the noise strength.

In ref. 3 we develop a third approach, based on construction of an explicit matrix representation of the stochastic evolution operator. The numerical implementation requires a truncation to finite dimensional matrices, and is less elegant than the smooth conjugation method, but makes it possible to reach up to order eight in expansion orders. As with the previous formulations, it retains the periodic orbit structure, thus inheriting valuable information about the dynamics.

The corrections to the escape rate were found to be a divergent series in the noise expansion parameter. This reflects that the corrections are calculated using the so-called cumulant expansion from other divergent quantities, the traces of the evolution operator $\mathscr{L}^n$.[3] In this article the focus is on the high order corrections for the special case of the first trace, $\mathrm{tr}(\mathscr{L})$. For the asymptotic study we have to calculate a sufficient number of corrections and here we are led naturally to a contour integral method. Next the asymptotic behaviour is extracted by the method of steepest descent.

2. THE STOCHASTIC EVOLUTION OPERATOR

In this section we introduce the noisy repeller and its evolution operator.

An individual trajectory in presence of additive noise is generated by iterating

$$x_{n+1} = f(x_n) + \sigma\xi_n \tag{1}$$

where $f(x)$ is a map, ξ_n a random variable with the normalized distribution $p(\xi)$, and σ parametrizes the noise strength. In what follows we shall assume that the mapping $f(x)$ is one-dimensional and expanding, and that the ξ_n are uncorrelated. A density of trajectories $\phi(x)$ evolves with time on the average as

$$\phi_{n+1}(y) = (\mathscr{L} \circ \phi_n)(y) = \int dx \, \mathscr{L}(y, x) \, \phi_n(x) \tag{2}$$

where the $\mathscr{L}$ evolution operator has the general form of

$$\begin{aligned} \mathscr{L}(y, x) &= \delta_\sigma(y - f(x)) \\ \delta_\sigma(x) &= \int \delta(x - \sigma\xi) \, p(\xi) \, d\xi = \frac{1}{\sigma} \, p\left(\frac{x}{\sigma}\right) \end{aligned} \tag{3}$$

For the calculations in this paper Gaussian weak noise is assumed. In the perturbative limit, $\sigma \to 0$, the evolution operator becomes

$$\mathscr{L}(x, y) = \frac{1}{\sqrt{2\pi} \, \sigma} \, e^{-(y - f(x))^2/2\sigma^2} \tag{4}$$

$$\sim \sum_{n=0}^{\infty} a_{2n} \sigma^{2n} \delta^{(2n)}(y - f(x)) \tag{5}$$

where $\delta^{(2n)}$ denotes the $2n$th derivative of the delta distribution and $a_{2n} = (2^n n!)^{-1}$ or $a_{2n} = m_{2n}/(2n)!$ for general noise with finite nth moment m_n. Strictly speaking we here only consider distributions with vanishing odd moments, but the formulas below can easily be generalized. The perturbative form of the evolution operator is obtained formally from (3) by Taylor expansion or by doing a saddle point integral discarding subdominant terms. In particular there may exist stationary points which do not correspond to periodic orbits. However, these will give exponentially small contributions and are therefore omitted in the weak noise limit $\sigma \to 0$. The map considered here is the same as in our previous papers, a quartic map on the $(0, 1)$ interval given by

$$f(x) = 20 \left[\frac{1}{16} - (\tfrac{1}{2} - x)^4\right] \tag{6}$$

3. RESIDUE METHOD FOR THE TRACE

The trace of the evolution operator in the weak-noise limit is obtained from

$$\mathrm{Tr} \, \mathscr{L} \sim \int dx \sum_{n=0}^{\infty} a_{2n} \sigma^{2n} \delta^{(2n)}(x - f(x)) \tag{7}$$

By expressing the delta function as

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \frac{1}{x - i\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) \tag{8}$$

we reformulate expression (7) as

$$\begin{aligned} \operatorname{Tr} \mathscr{L} \sim \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{(2n)!\, a_n \sigma^{2n}}{2\pi i} \bigg[\int dx \, \frac{1}{(x - f(x) - i\varepsilon)^{2n+1}} \\ - \int dx \, \frac{1}{(x - f(x) + i\varepsilon)^{2n+1}} \bigg] \end{aligned} \tag{9}$$

obtaining just the result as in [7, Section 5.3, Eq. (19), p. 119].

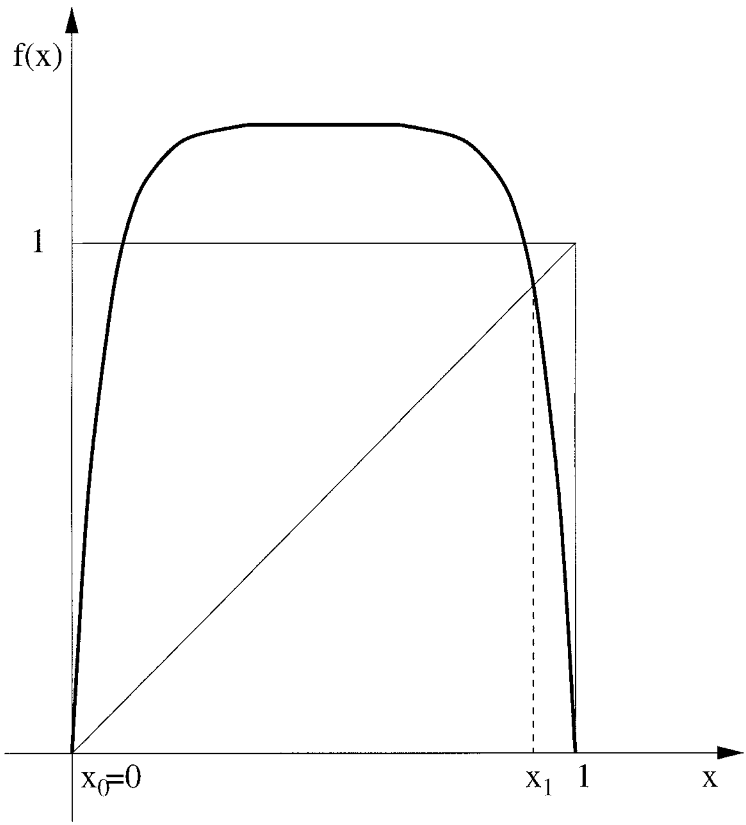

Fig. 1. The map on the [0, 1] interval.

The integrals in (9) have poles close to the fixpoints of the map, which are defined by

$$x^* - f(x^*) = 0 \tag{10}$$

For the map in question, a quartic, there are four points in the complex plane which satisfy this equation: two of them are on the real axis: one at $x_0 = 0$, the other x_1 close to unity, as it can be seen on Fig. 1. The two real fix points correspond to orbits 0 and 1 in the language of symbolic dynamics.[6] The integral in (9) containing $-i\varepsilon$ in the denominator has a pole at x_0 shifted down from the real axis into the negative imaginary half plane and a pole at x_1 shifted up into the positive imaginary half plane. At the other integral in (9) the poles are moved in the opposite direction.

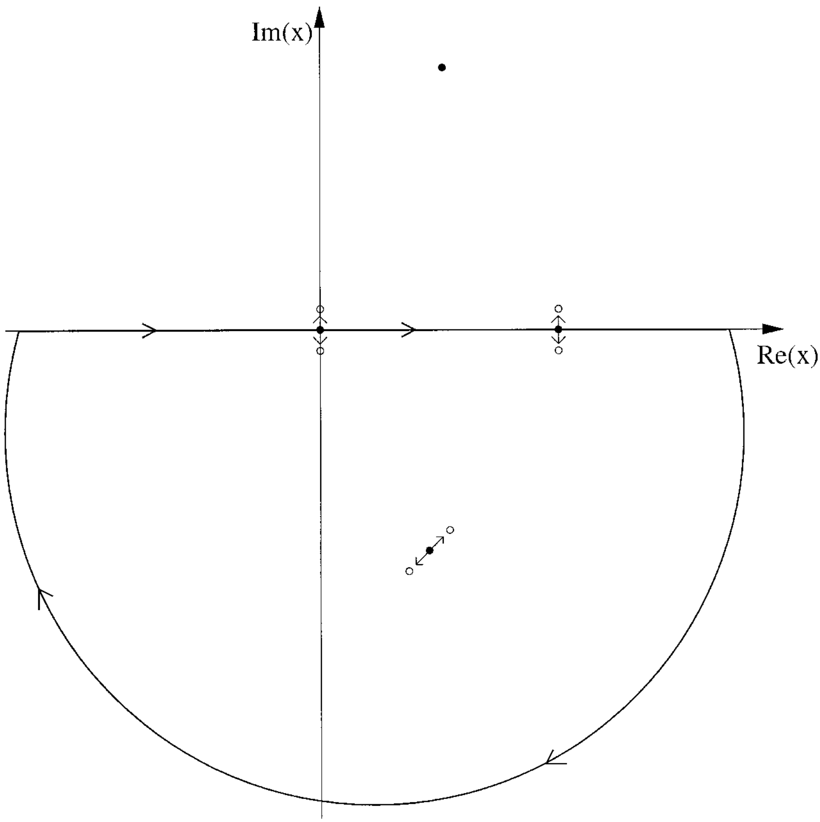

Fig. 2. The chosen contour and the motion of poles.

We shall now evaluate the integrals by contour integration. The chosen contour is shown on Fig. 2. The integral containing the $-i\varepsilon$ will pick up contribution from the fixpoint at $x=0$, the other integral from the fixpoint at x_1. Both will pick up a contribution from the complex fixpoint, but they cancel out in the $\varepsilon \to 0$ limit. The result of the integration is

$$\operatorname{Tr} \mathscr{L} \sim -\sum_{n=0}^{\infty} \frac{(2n)!\,\sigma^{2n}}{2^n n!}\left[\operatorname{Res} F_n(x_0) - \operatorname{Res} F_n(x_1)\right] \tag{11}$$

where

$$F_n(x) = \frac{1}{(x - f(x))^{2n+1}} \tag{12}$$

We used this method to calculate further corrections (around σ^{70}) to the trace for the orbit 0 and orbit 1. This exact method justified the previous approximate calculations to all known digits. It turns out that the corrections for orbit 0 all were positive whereas those for orbit I involved sign changes with an apparent period. Furthermore both were asymptotic series.

4. HIGHER ORDER NOISE CORRECTIONS

Practical reasons restricted the calculations with the residue method to around order seventy in the expansion parameter. Nevertheless it is possible to give approximate statements for even higher order.

First we transform the residue integrals in (9) as

$$\begin{aligned} I_n &= \frac{(2n)!\,a_{2n}}{2\pi i} \oint dx \, \frac{1}{(x - f(x) \mp i\varepsilon)^{2n+1}} \\ &= \frac{(2n)!\,a_{2n}}{2\pi i} \oint dx \, e^{(-(2n+1)\log(x - f(x) \mp i\varepsilon))} \end{aligned} \tag{13}$$

Using n as a large parameter the integral is next calculated by the method of steepest descent. The initial loops, $\mathscr{C}_0$ and $\mathscr{C}_1$, for each fixpoint are expanded until the paths of steepest descents are reached. We, refer to Fig. 3.

The paths of steepest descent pass through the saddle points satisfying

$$\frac{d}{dx}\log(x_s - f(x_s) \mp i\varepsilon) = \frac{1 - f'(x_s)}{x_s - f(x_s) \mp i\varepsilon} = 0, \qquad f'(x_s) = 1 \tag{14}$$

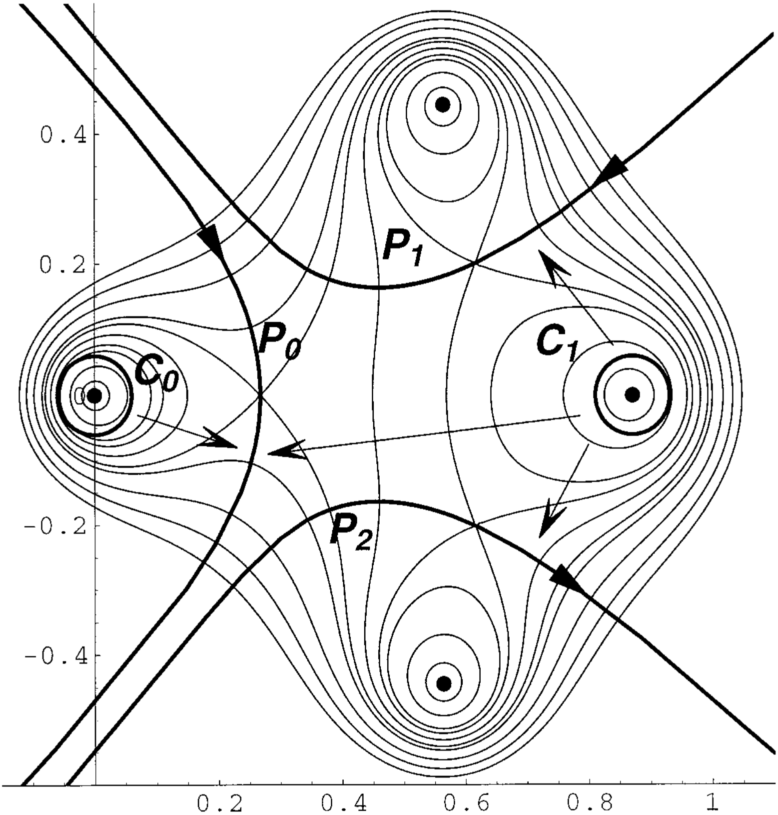

Fig. 3. Contourlines of $|x - f(x)|$. Filled circles indicate fixpoints of the map, crossing contours correspond to saddles and thick lines show paths of steepest descent.

Around each saddle we calculate the leading contribution and its corrections as usual. In the following we will divide the discussion into orbit 0 and orbit 1. For our map there are three saddle points: one real and two complex conjugate.

In case of orbit 0, the integral is carried out on the infinite path $\mathscr{P}_0$ shown on figure Fig. 3. Here only the saddle point on the real axis contributes.

The result of the integration to leading order is

$$I_n^0 \sim \frac{(2n)!}{2^{n+1} n!\, \pi}\, e^{-(2n+1)\log(x_s - f(x_s))} \sqrt{\frac{\pi(x_s - f(x_s))}{(2n+1)\, f''(x_s)}} \tag{15}$$

For a convenient representation we introduce the factors

$$C_0 := \frac{1}{2\pi\sqrt{2(x_s - f(x_s))\, f''(x_s))}} \tag{16}$$

$$b_0 := \frac{2}{(x_s - f(x_s))^2} \tag{17}$$

and the result takes the form

$$I_n^0 \sim C_0 \frac{b_0^n \Gamma(n+1/2)}{\sqrt{n+1/2}} \tag{18}$$

We show the ratio of consecutive terms I_{n+1}/I_n on Fig. 4 with the ratios from the exact residue calculation and conclude that the real saddle indeed controls the leading behaviour.

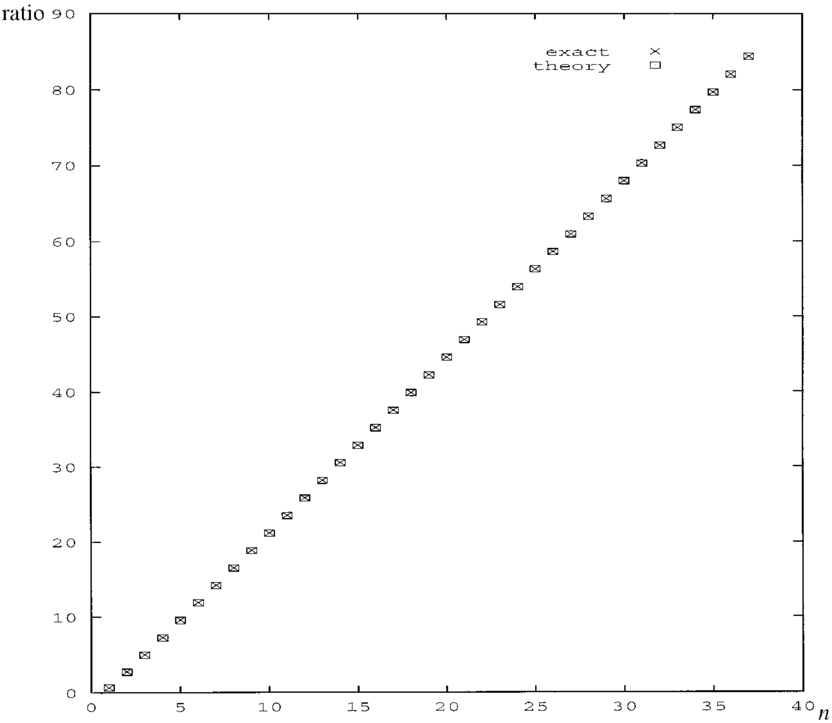

Fig. 4. The ratio of the noise corrections I_{n+1}/I_n for orbit 0.

For the orbit 1 all three saddles are relevant, since the initial loop $\mathscr{C}_1$ decomposes into $\mathscr{P}_0$, $\mathscr{P}_1$ and $\mathscr{P}_2$. However, the real saddle now becomes subdominant as can be seen from e.g., the contour plot Fig. 3. By analyticity of the reap the two complex saddles give results that are complex conjugate to each other.

At the complex saddle points (z_1, z_2) we introduce similar factors C_1, b_1 as in (16, 17) now complex.

The contribution from these two complex saddles to I_n^1 call now be written as

$$\frac{\Gamma(n+1/2)}{\sqrt{n+1/2}}\,[C_1 b^n{}_1 + C_1^*(b_1^*)^n] = \frac{\Gamma(n+1/2)}{\sqrt{n+1/2}}\,[rq^n \cos(n\phi - \alpha)] \quad (19)$$

Here r, q, ϕ and α are easily found from C_1, b_1 above.

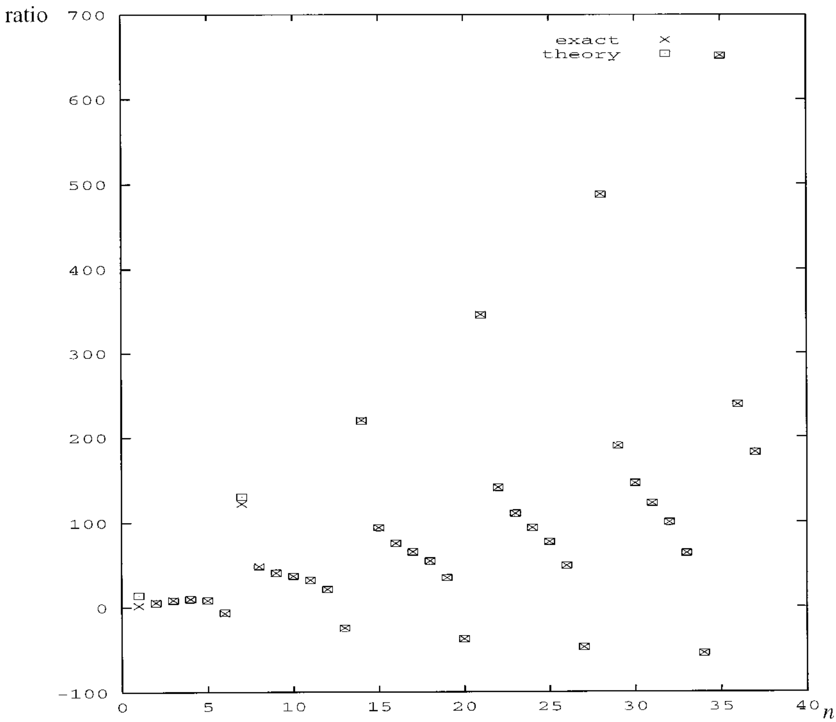

Fig. 5. The ratio of the noise corrections I_{n+1}/I_n for orbit 1.

The final result for orbit 1 also includes the real saddle and to leading order is given by

$$I_n^1 \sim \frac{\Gamma(n+1/2)}{\sqrt{n+1/2}}\left[C_0 b_0^n + r q^n \cos(n\phi - \alpha)\right] \tag{20}$$

We remark that the same calculation could be done for another distribution than Gaussian. One would find an identical functional form up to a multiplicative constant, the corresponding moment.

Here the plot of the ratios of the consecutive terms Fig. 5 show a more complex behaviour than for orbit 0. This we attribute to the presence of dominant complex saddles, which will introduce, phases and hence the sign changes already mentioned. The apparent periodicity of seven is nicely captured by the two complex saddles. Nevertheless, high order calculations actually find deviations from this pattern in agreement with that, the phase calculated does not evaluate exactly to an integer multiple of $2\pi/7$. The saddle point expansion in general improves for high orders as can be seen on

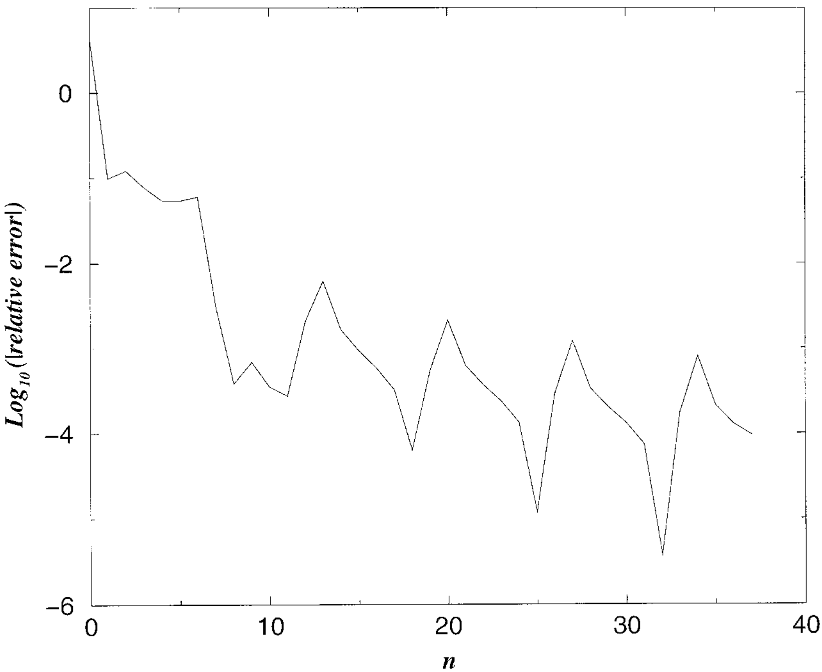

Fig. 6. The logarithm of the relative error $\mathrm{Log}_{10}(|(I_n^{(\mathrm{theory})} - I_n^{(\mathrm{exact})})/I_n^{(\mathrm{exact})}|)$ of the noise corrections for orbit 1.

Fig. 6. For our calculations two corrections to the complex saddles give the best results in the regime of n we are considering. This is because as an asymptotic series truncation is required at an optimal point when n is finite. So adding more corrections will only improve the results for higher n but actually make the lower less accurate.

5. SUMMARY AND CONCLUSION

We have studied the evolution operator for a discrete, Langevin equation with a strongly hyperbolic classical dynamics and noise with finite moments. Using a perturbative expansion of the evolution operator we have calculated high order corrections to its trace in the case of a quartic map and Gaussian noise. We have found the asymptotic behaviour of the corrections using the method of steepest descent. The functional form of the high order corrections have the same form independent of the actual noise distribution up to a constant given by the corresponding moment. The asymptotics of the trace of the evolution operator are governed by subdominant terms corresponding to terms previously neglected in the perturbative expansion.

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